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Investigating the integrability of the Lyness mappings

B Grammaticos[1], **A Ramani**[2] **and T Tamizhmani**[3]

[1] IMNC, Université Paris VII-Paris XI, CNRS, UMR 8165, Bât. 104, 91406 Orsay, France
[2] Centre de Physique Théorique, Ecole Polytechnique, CNRS, 91128 Palaiseau, France
[3] Department of Mathematics, Kanchi Mamunivar Centre for Postgraduate Studies, Puducherry, India

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Abstract
We examine a family of difference equations, known as the Lyness mappings, from the point of view of integrability. We show that the mappings satisfy two different integrability criteria and are thus good integrability candidates. We introduce an ansatz which reduces the mappings to bilinear form and we show that the equations obtained are just reductions of the Hirota–Miwa equation which establishes the integrable character of the Lyness mappings. Finally, we discuss the construction of explicit invariants for some instances of the mapping.

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1. Introduction

The investigation of the integrable character of a given system, be it continuous or discrete, usually comprises two phases. The first one is essentially exploratory. One examines the system for integrability by testing whether known integrability criteria are satisfied. Sometimes this is complemented by a numerical study of the behaviour of the system where smooth behaviour constitutes an indication of integrability. If the first phase yields a positive result and the system may be considered a serious candidate for integrability one proceeds to the second phase of the investigation. The latter consists in trying to establish, usually through a constructive approach, the integrable character. In some cases this means integrating the system to some equation the integrability of which is already established. In other cases, it may be sufficient to show that the system at hand is a reduction of some more general known integrable system. Once the process has been completed with success one can at last claim to have derived a new integrable system.

In the case of discrete systems, like the one that will be studied in this paper, the detection of integrability usually proceeds through two well-established criteria. The first one, known under the name of singularity confinement [1], is based on a property characterizing all systems integrable by spectral methods, namely, that any singularity spontaneously appearing (due to the choice of initial conditions) disappears after a few iterations steps. The second one is often referred to as 'algebraic entropy' [2]. Its proposal formalizes the observation by Arnold [3] and Veselov [4] that the integrability of discrete systems is related to the slow growth of some characteristic quantity. The precise algorithm, due to Viallet and collaborators [2, 5], concerns rational mappings and computes the homogeneous degree of the numerator and denominator of the iterates. An exponential increase is an indication of nonintegrability while integrable mappings have a degree which grows polynomially with the number of iterations.

In this paper we shall consider the Lyness mappings [6, 7] which are usually presented under the form

$$x_{n+N}x_n = a + x_{n+1} + x_{n+2} + \cdots + x_{n+N-1}. \tag{1.1}$$

A form more convenient for some of the calculations we shall perform is the discrete derivative of (1.1):

$$x_{n+N}(1 + x_n) = x_{n+1}(1 + x_{n+N+1}). \tag{1.2}$$

We will start by applying the two aforementioned discrete integrability criteria and show that the Lyness mappings satisfy both of them. We will then proceed to bilinearize the mappings and show that they are reductions of the Hirota–Miwa (discrete KP) equation [8, 9], thus establishing their integrability. Finally, we will examine the integrals of the bilinear Lyness mappings for low values of N based on recent results of Maruno and Quispel (MQ) [10], and their relation to known invariants of (1.1).

2. Integrability criteria applied to the Lyness mapping

We shall start our investigation with the simplest nontrivial case, namely $N = 2$

$$x_{n+2}x_n = a + x_{n+1}. \tag{2.1}$$

This mapping is a well-known integrable one, being a member of the QRT [11] family. In order to apply the singularity confinement we start from a finite value for x_n and choose $x_{n+1} = -a$. We obtain the following pattern $\{0, -1, \infty, \infty, -1, 0\}$ and the subsequent xs are finite. Thus the singularity of (2.1) is confined, as expected.

Next we examine the case $N = 3$:

$$x_{n+3}x_n = a + x_{n+1} + x_{n+2}, \tag{2.2}$$

i.e. we start from finite values for x_n, x_{n+1} and choose $x_{n+2} = -a - x_{n+1}$. We find the following singularity pattern $\{0, -1, f_1, \infty, \infty, g_1, -1, 0\}$, where f_1 and g_1 are two finite expressions involving x_n, x_{n+1} and a. Again the singularity is confined.

We have obtained the singularity patterns for several more values of N. In every case we found a confined singularity corresponding to the following pattern:

$$\{0, -1, f_1, f_2, \ldots, f_{N-2}, \infty, \infty, g_1, g_2, \ldots, g_{N-2}, -1, 0\}.$$

We surmise that this holds for all values of N, which would constitute a first indication of the integrability of this mapping.

The second criterion we are going to apply is that of low growth. We found that the easiest way to implement it is the one introduced by Halburd under the name of Diophantine integrability [12]. Halburd considers the iterates of the mapping starting from rational initial conditions (and rational values of the parameters, if any) and introduces the 'height' $H(x)$ of an element x defined as $H = \max(p, q)$ where $x = p/q$ (with p/q irreducible). The growth of $H(x_n)$ with n is a measure of the complexity of the mapping. In other words too fast a growth of $H(x)$ is considered incompatible with integrability. More precisely, the

Diophantine integrability introduced by Halburd requires that the logarithmic height of iterates $h(x_n) = \log H(x_n)$ grows no faster than a polynomial in n.

We have performed several numerical experiments on Lyness mappings for values of N up to 20. We have started from integer initial conditions and some integer values for the parameter a. Given the simplicity of the calculations several hundred of iterations can be easily performed. The calculations are carried in rational arithmetic, the only simplification being the factoring out of the greatest common divisor of numerator and denominator. We have studied the limit h_n/n^2 when $n \to \infty$ and found that for all Lyness mappings this ratio converges to a finite number. This is another indication of integrability since it tells us that the degree growth of the mapping is quadratic. The reason of this quadratic growth, independent of the order of the mapping, i.e. of the value of N, will become apparent in the following section.

The slow growth obtained in our calculations disappears immediately when we perturb the mapping (for instance, by taking the coefficient of one of the terms in the summation on the right-hand side of (1.1) different from unity). In this case, a significant growth index is $\log h_n/n$. We have found that in the perturbed case this quantity converges to a finite value indicating thus an exponential growth of the complexity of the iterates.

3. Bilinearizing the Lyness mapping

In the previous section, we have seen that the Lyness mappings satisfy two fundamental discrete integrability criteria. They are thus excellent integrability candidates. In order to show that the Lyness mappings are indeed integrable we shall make use of the bilinear formalism.

The first step towards the bilinearization of a given system is an ansatz where the nonlinear variable, here x, is expressed in terms of τ-functions. In [13], we have shown how one can use the information from the singularity structure of the solutions in order to propose the adequate ansatz. Moreover a schematic singularity pattern, like the one presented in section 2, suffices. We also surmised that the number of τ-functions is related to the number of different singularity patterns. Oversimplifying the situation we may say that if only one singularity pattern exists, a single τ-function may suffice. Moreover, since τ-functions are entire, the nonlinear variable must be expressed in terms of ratios of products of such functions. Since for the Lyness mapping x assumes the values of 0 and -1, and diverges at precise positions the following ansatz may be proposed:

$$x_n = A_n \frac{\tau_{n-N}\tau_{n+N+1}}{\tau_n \tau_{n+1}} = -1 - B_n \frac{\tau_{n-N+1}\tau_{n+N}}{\tau_n \tau_{n+1}}. \tag{3.1}$$

Through a gauge transformation one can set B to one without loss of generality. It is elementary to check that this reproduces the singularity pattern of the Lyness mapping. Moreover, substituting (3.1) in the appropriate combination, into the form (1.2), one verifies that the discrete derivative of the Lyness mapping is identically satisfied, provided that A_n is periodic of period $N - 1$.

Equating the last two terms of (3.1) we find the bilinear equation that τ should obey

$$A_n \tau_{n-N}\tau_{n+N+1} + \tau_{n-N+1}\tau_{n+N} + \tau_n \tau_{n+1} = 0 \tag{3.2}$$

i.e. the bilinear form of the Lyness mappings. The interesting result here is that, for every N, (3.2) is just a reduction of the Hirota–Miwa equation. This is obvious when A is a constant and also when A is periodic with the right period, as will be discussed below.

The Hirota–Miwa is the discrete analogue of the KP equation. It is usually given in the form:

$$A\tau_{k-1,l,m}\tau_{k+1,l,m} + B\tau_{k,l-1,m}\tau_{k,l+1,m} + C\tau_{k,l,m-1}\tau_{k,l,m+1} = 0 \tag{3.3}$$

or, equivalently,

$$A\tau_{k,l,m+1}\tau_{k+1,l+1,m} + B\tau_{k,l+1,m}\tau_{k+1,l,m+1} + C\tau_{k+1,l,m}\tau_{k,l+1,m+1} = 0 \tag{3.4}$$

where, *a priori*, A, B and C are constants. In what follows we will work with the form (3.4). In order to retrieve (3.2) from it we perform a one-dimensional reduction. Starting from the triplet (k, l, m) we introduce a single index n through $n = k + Nl - Nm$. Provided $\tau_{k,l,m}$ depends only on n, we obtain precisely the bilinear form of the Lyness mapping with $B = C = 1$ and A constant. However, a gauge transformation allows us to choose for A_n any function of period $N - 1$. Indeed, if we consider the full three-dimensional equation (3.4) one can use *any* function Ω of the three variables k, l, m as gauge function $\tau_{k,l,m} = \Omega_{k,l,m}\theta_{k,l,m}$, obtaining for θ an equation of the same form as (3.4) but with coefficients that are now expressed in terms of Ω. The problem of what choices of Ω are compatible with the one-dimensional reduction is not extremely difficult but we do not even need to solve it in general. All we need is an explicit expression that works for the present case. So let us take $\Omega(k, l, m) = F(k + l - m)$. Starting from (3.4) for τ with $B = C = 1$ we get

$$\begin{aligned} &AF(k+l-m-1)F(k+l-m+2)\theta_{k,l,m+1}\theta_{k+1,l+1,m} \\ &\quad + F(k+l-m)F(k+l-m+1)\theta_{k,l+1,m}\theta_{k+1,l,m+1} \\ &\quad + F(k+l-m)F(k+l-m+1)\theta_{k+1,l,m}\theta_{k,l+1,m+1} = 0. \end{aligned} \tag{3.5}$$

Dividing by $F(k+l-m)F(k+l-m+1)$ we recover the same form as (3.4) with the constant A becoming now a free function $\tilde{A}$ of $k + l - m$ only, as one can solve for F whatever $\tilde{A}$ is. We now want (3.5) to be reducible to (3.2). This means that $\tilde{A}(k + l - m)$ must depend on $n = k + Nl - Nm$ only. If two sets of integers k_1, l_1, m_1 and k_2, l_2, m_2 are such that $n_1 \equiv k_1 + Nl_1 - Nm_1$ and $n_2 \equiv k_2 + Nl_2 - Nm_2$ are equal, then the difference of the arguments of $\tilde{A}$ at these two points is $(N-1)(l_2+m_1-m_2-l_1)$, and thus the respective values of $\tilde{A}$ coincide provided that $\tilde{A}$ has period $(N - 1)$. This shows that we can indeed obtain (3.2) with $\tilde{A}$ any periodic function of period $(N - 1)$ through this gauge transformation.

The relation of the Lyness mappings to the Hirota–Miwa equation establishes the integrable character of the former: they are reductions of an integrable partial difference equation. This relation explains also the quadratic complexity growth obtained in the previous section. The Hirota–Miwa equation has quadratic growth for the degree of the iterates of a given initial condition and thus the Lyness mappings, being a reduction of Hirota–Miwa, can only have quadratic growth independently of their order, i.e. for all values of N.

4. On the invariants for the Lyness mapping

Since the Lyness mappings are just reductions of the Hirota–Miwa equation it is clear that their invariants can be deduced from the invariants of the latter. However, a difficulty exists: the invariants of the Hirota–Miwa equation are not known in full generality. In a recent paper [10], Maruno and Quispel have proposed an approach for the construction of the conservation laws of the Miwa and the Hirota–Miwa equations but they have given explicitly only a few cases without presenting the general solution. Still these results are most useful, as we shall see in what follows. The question of invariants of the Lyness mapping was also addressed by Bastien and Rogalski [14] who presented explicit forms of a few such quantities. Still, despite these advances the question remains open. In this section, we shall see how the results from the bilinear formalism can be compared to those obtained directly in the nonlinear approach for the first, low N, Lyness mappings.

Before examining specific cases, let us summarize what is known and start from a basic observation. In section 1 we have presented two forms of the Lyness mappings (1.1) and

(1.2), the second being the discrete derivative of the first. Given this relation it is clear that the quantity a appearing in the form (1.1) is a constant of integration of the form (1.2). Thus, when one works with the derivative form (as is the case for the bilinear expression (3.2)) one should expect one extra invariant related to the constancy of a. Two invariants are known for all values of N. They can be written as

$$G = \frac{(1+x_n)(1+x_{n+1})\cdots(1+x_{n+N})}{x_{n+1}\cdots x_{n+N-1}} \tag{4.1}$$

and

$$H = \frac{(1+x_n+x_{n+1})(1+x_{n+1}+x_{n+2})\cdots(1+x_{n+N-1}+x_{n+N})}{x_{n+1}\cdots x_{n+N-1}}. \tag{4.2}$$

Using the bilinear ansatz (and taking a strictly constant A) we can show that the first invariant is in fact $G = (-1/A)^{N-1}$. A third invariant was obtained by Gato *et al* [15], for N odd, larger or equal to 5, but we are not going to write it explicitly here.

The case $N = 2$ is particularly simple. The existence of the two invariants, G and H, does not create difficulties. First, they correspond to the derivative form (1.2) rather than (1.1). Moreover, in the $N = 2$ case, these two invariants are not independent. A straightforward calculation shows that $G - H = a - 1$, where a is the parameter appearing in (1.1). Moreover, the mapping (1.1) is a member of the QRT family and its invariant

$$I = x_n + x_{n+1} + \frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_n} + (a+1)\left(\frac{1}{x_{n+1}} + \frac{1}{x_n}\right) + \frac{a}{x_n x_{n+1}} \tag{4.3}$$

leads to its integration in terms of elliptic functions. A simple calculation shows that $I = H - 3$ where it is understood that one must use (2.1) in order to eliminate x_{n+2} in terms of a. Turning now to the bilinear formalism one sees immediately that the mapping (3.2) for $N = 2$ is a fifth-order one and thus needs extra invariants which involve the τ-function and cannot be expressed in terms of x. Using the results of Maruno and Quispel it is indeed possible to construct such invariants. We find for instance that

$$\Lambda = \frac{\tau_{n+1}^2}{\tau_n\tau_{n+2}} + \frac{\tau_n^2}{\tau_{n+1}\tau_{n-1}} + \frac{\tau_{n-1}^2}{\tau_n\tau_{n-2}} + \frac{\tau_{n+1}\tau_{n-1}}{\tau_{n-2}\tau_{n+2}} - A\frac{\tau_{n+2}\tau_{n-2}}{\tau_{n-1}\tau_{n+1}} \tag{4.4}$$

in a conserved quantity of (3.2) for $N = 2$ which cannot be expressed in terms of x. Indeed, given the expression of x in terms of the τ-function (3.1) one sees that there exists a gauge of the latter which leaves x invariant: one can multiply the even-index τ-functions by a constant k and the odd-index ones by $1/k$. Any invariant given in τ-functions which can be expressed in terms of x must be invariant under this gauge. This is *not* the case for Λ and the use of the gauge makes it possible to split it into two parts (scaling like k^2 and $1/k^2$ respectively). We have $\Lambda = J + L$ where $J = \left(\tau_n^2 - A\tau_{n+2}\tau_{n-2}\right)/(\tau_{n-1}\tau_{n+1})$ and $L = \tau_{n+1}^2/(\tau_n\tau_{n+2}) + \tau_{n-1}^2/(\tau_n\tau_{n-2}) + \tau_{n+1}\tau_{n-1}/(\tau_{n-2}\tau_{n+2})$. Following this logic we expect the quantities J and L to be interchanged at each step. Therefore, their product JL should also be an invariant (a straightforward calculation shows that this is indeed the case). Moreover, it should be possible to express this last conserved quantity in terms of the variable x, since it is invariant under the aforementioned gauge.

What is particularly interesting in the $N = 2$ case are the interrelations one can establish between various mappings of QRT type. Indeed, starting from the invariant G and assuming that its conserved value is g one finds that the quantity $y = x + 1$ obeys the mapping

$$y_{n+1}y_{n-1} = g\frac{y_n - 1}{y_n}. \tag{4.5}$$

It is interesting to point out here that the JL invariant mentioned in the previous paragraph has a particularly simple expression in terms of y. We find

$$JL = \frac{1}{y_n y_{n+1}} - \frac{1}{y_n} - \frac{1}{y_{n+1}} + A(y_n + y_{n+1} + 1). \tag{4.6}$$

As a matter of fact (4.6) is just the QRT invariant of the mapping (4.5) given that A and the conserved value g of G, are related through $gA = -1$. It is interesting to point out that, expressed in terms of x, the QRT invariant JL can also be expressed in terms of the basic ones. The relation can be written simply as

$$JL = \frac{a-3}{g} - 1. \tag{4.7}$$

Similarly, starting from the invariant H with conserved value h and introducing $z = x + 1/2$ we find the mapping

$$(z_{n+1} + z_n)(z_n + z_{n-1}) = h(z_n - 1/2). \tag{4.8}$$

Its QRT invariant is simply G, rewritten as

$$G = (z_{n+1} + 1/2)(z_n + 1/2)\left(\frac{h}{z_{n+1} + z_n} - 1\right). \tag{4.9}$$

We see that, thanks to the underlying QRT structure, the Lyness $N = 2$ mapping (2.1) and mappings (4.5) and (4.8) are intimately related.

We turn now to the case $N = 3$. Starting from (1.2) we obtain

$$\frac{(1+x_n)(1+x_{n+2})}{x_{n+1}} = \frac{(1+x_{n+2})(1+x_{n+4})}{x_{n+3}}, \tag{4.10}$$

which means that the quantity $(1+x_n)(1+x_{n+2})/x_{n+1}$ is a (parity-dependent) constant. We are thus led to the introduction of two invariants

$$K = \frac{(1+x_n)(1+x_{n+2})}{x_{n+1}} \tag{4.11}$$

and

$$M = \frac{(1+x_{n+1})(1+x_{n+3})}{x_{n+2}}. \tag{4.12}$$

The invariant G is thus simply $G = KM$, while some elementary calculations allow us to show that H is not independent but can be expressed as a combination of K, M and a. As in the $N = 2$ case, one can show that more invariants exist involving the τ-function which cannot be expressed in terms of x as they are not invariant under the gauge as explained above. Moreover, we can rewrite (4.11) and (4.12) as a mapping, introducing $y = x + 1$

$$y_{n+1} y_{n-1} = p(y_n - 1), \tag{4.13}$$

where p is an even–odd depending parameter, i.e. $p_n = q + r(-1)^n$.

Finally, we have examined the case $N = 4$. This case is of interest because its complete integration would necessitate three invariants and only two, G and H, are known. The bilinear approach applied to this case, using the results of (MQ), does not allow us to surmount this difficulty: no extra invariant involving the nonlinear variable x was found (though we did find invariants in τ which cannot be written in terms of x).

5. Conclusions

In this paper, we have examined the Lyness mappings from the integrability point of view. Since the integrability of the first members of the family had already been established it was natural to speculate on the possible integrability of the entire family. We have thus started by applying two well-known integrability criteria on several (admittedly of low N, but well beyond the known integrable ones) members of the Lyness family. The answer was unambiguous: all of them satisfied both integrability criteria. Thus, it was reasonable to expect the Lyness mappings to be integrable in all generality.

The integrability of the Lyness mappings was established thanks to the bilinear approach. We have indeed shown that, by introducing the suitable ansatz and casting the mappings into a bilinear form, it was possible to establish the fact that they are reductions of the Hirota–Miwa (discrete KP) equation for all values of the parameter N. Based on this relation it would have been possible to explicitly construct the invariants if the Lyness mappings in full generality be it not for a minor (major?) problem: the invariants of the Hirota–Miwa equation are not fully known. This in fact indicates a possible axis of research, namely to attempt the construction of the conservation laws of the Hirota–Miwa equation in full generality. Once the latter are obtained the derivation of the Lyness mappings invariants will be reduced to the level of a simple exercise.

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References

[1] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825
[2] Hietarinta J and Viallet C M 1998 *Phys. Rev. Lett.* **81** 325
[3] Arnold V I 1990 *Bol. Soc. Bras. Mat.* **21** 1
[4] Veselov A P 1992 *Commun. Math. Phys.* **145** 181
[5] Bellon M P and Viallet C-M 1999 *Commun. Math. Phys.* **204** 425
Bellon M P, Maillard J-M and Viallet C-M 1991 *Phys. Rev. Lett.* **67** 1373
[6] Bastien G and Rogalski M 2004 *J. Diff. Eqns Appl.* **10** 977
[7] Cima A, Gasull A and Mañosa V 2007 *J. Diff. Eqns Appl.* **13** 855
[8] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3785
[9] Miwa T 1982 *Proc. Japan Acad.* **58** 9
[10] Maruno K and Quispel G R W 2006 *J. Phys. Soc. Japan* **75** L-123001
[11] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* D **34** 183
[12] Halburd R G 2005 *J. Phys. A: Math. Gen.* **38** L263
[13] Ramani A, Grammaticos B and Satsuma J 1995 *J. Phys. A: Math. Gen.* **28** 4655
[14] Bastien G and Rogalski M 2008 Results and conjectures about order q Lyness' difference equation
[15] Gao M, Kato Y and Ito M 2004 *Appl. Math. Lett.* **17** 1183